Malaysian Journal of Science **18**: 51-53 (1999)

A note on the generalised Lomax distribution as a lifetime model

S.H. Ong and E.K. Ong

Department of Mathematics, University of Malaya, 50603 Kuala Lumpur, Malaysia

Abstract. The behaviour of the hazard rate of the generalised Lomax distribution is examined. The hazard rate is either decreasing or upside-down bathtub shaped. An example of a goodness-of-fit to lifetime data is given.

Abstrak. Perangai kadar mudarat taburan am Lomax dikaji. Mudarat adalah menurun atau berbentuk tab mandi terbalik. Satu contoh data *goodness-of-fit* ke masa hayat diberikan.

Introduction

The generalized Lomax (GL) distribution [1,2] has probability density function (pdf)

$$f(x)=\frac{\beta^{\alpha}x^{\nu-1}}{B(\nu,\alpha)(\beta+x)^{\alpha+\nu}}, \qquad \alpha,\nu,x>0, \tag{1,1}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The GL distribution is related to the F distribution as follows. If the random variable (rv) X has the GL distribution, then rv $Z = \alpha x/(\nu\beta)$ is an F rv with 2ν and 2α degrees of freedom. Although the F distribution has been considered as a failure time model (see, for instance [3], p. 28), basic survival or reliability properties do not seem to have been discussed in detail in the literature. This note considers the behaviour of its hazard rate (failure rate) and gives an example of fit to lifetime data.

The GL distribution may be expressed as a mixed gamma distribution

$$f(x)=\int_0^{\infty}\frac{\lambda^{\nu}x^{\nu-1}e^{-\lambda x}}{\Gamma(\nu)}g(\lambda)d\lambda \tag{1.2}$$

where $g(\lambda) = \lambda^{\alpha}\beta^{\nu-1}e^{-\nu\beta}/\Gamma(\alpha)$ is the gamma pdf. If $\nu = 1$, this gives the Pareto Type 2 or Lomax distribution [2], or the single-hit distribution considered by Turner [4], Lefante and Inman [5] and others. Mixed gamma distributions and their reliability properties have been examined by Ong [6].

Let $f(x)$ denote the failure time pdf. The hazard rate function is defined as

$$h(x) = f(x)/R(x)$$

where $R(x) = \Pr(X>x)$ is the reliability (survival) function. The GL has a decreasing hazard rate function (DLIR) for $\nu > 1$, and this function is upside-down bathtub (UBT) shaped for $\nu > 1$. UBT-shaped hazard rate functions are found in situations like accelerated life testing. Two well-known lifetime models, the lognormal and inverse-Gaussian [7], have a UBT-shaped hazard rate function. The following result of Glaser [8] will be used to prove that the GL distribution has a UBT-shaped hazard rate.

Assume that the failure time pdf $f(t)$ is defined, positive, continuous and twice differentiable on $(0, \infty)$. Let

$$g(t)=1/h(t)$$

and

$$g'(t) = g(t)\eta(t) - 1$$

where $\eta(t) = -f'(t)/f(t)$.

Theorem [8]. Suppose there exists $t_o > 0$ such that

$$\eta'(t) > 0 \text{ for all } t \in (0, t_0),\ \eta'(t_0) = 0$$

and

$$\eta'(t) < 0 \text{ for all } t > t_0. \quad (1.3)$$

(i) If there exists $y_0 > 0$ such that $g'(y_0) = 0$, then $h(t)$ is UBT shaped.

(ii) If there does not exists $y_0 > 0$ such that $g'(y_0) = 0$, then $h(t)$ is DHR.

Instead of using (i) and (ii) in the above theorem, the following modified approach may be employed ([8], p. 669):

Suppose $L = \lim f(t)$ exists and equals 0 or ∞.

(i) If (1.3) holds and $L = 0$, then $h(t)$ is UBT shaped.

(ii) If (1.3) holds and $L = \infty$, then $h(t)$ is DHR.

An application of the GL distribution to reliability data is considered. A comparison of the hazard rates of the GL and inverse Gaussian distributions for this data set is given.

Hazard rate of the Generalized Lomax Distribution

The gamma distribution with shape parameter $\nu \le 1$ is DHR. Since the mixture of a DIIR distribution is again DHR [9], the GL distribution given as a mixed gamma by (1.2) is a DHR distribution. When $\nu > 1$, the GL distribution has a hazard rate that increases and then decreases (upside-down bathtub shaped).

Theorem. The generalized Lomax distribution with pdf (1.1) has a hazard rate that is upside-down bathtub shaped for $\nu > 1$.

Proof. Let $\nu > 1$. To prove the theorem, we use the results of Glaser given in the introduction. Consider

$$\eta(x) = -f'(x)/f(x) \text{ and } \eta'(x) = d\eta(x)/dx.$$

For the GL distribution,

$$\eta(x) = [x(\alpha + \nu) - (x + \beta)(\nu - 1)]/x(x + \beta),$$

$$\eta'(x) = [(x + \beta)^2(\nu - 1) - (\alpha + \nu)x^2]/[x(x + \beta)]^2.$$

If there exists $x_0 > 0$ such that $\eta'(x) > 0$ for all $x \in (0, x_0)$, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $x > x_0$, and $\lim f(x) = 0$, then $f(x)$ has a upside-down bathtub-shaped hazard rate.

Take $x_0 = \beta\sqrt{(\nu - 1)}/[\sqrt{(\alpha + \nu)} - \sqrt{(\nu - 1)}]$ with

$\alpha > 0$, $\nu > 1$.

Then $\eta'(x) > 0$ in $(0, x_0)$, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ in (x_0, ∞). Furthermore, $\lim f(x) = 0$. The conclusion follows.

An application to data analysis

The GL distribution is fitted to the repair time (hours) data for an airborne communication transceiver [10], p. 156: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8 ,0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Maximum likelihood (ML) estimation of the GL parameters is carried by using the numerical minimization routine NELMIN [11,12]. The random starting point procedure has been used to ensure with a fairly high probability that the global maximum is attained. The estimates (β, α, ν) corresponding to the largest function value for these different starting points are chosen as the ML estimates. The ML estimates are found to be $(\beta, \alpha, \nu) = (0.45, 1.38, 4.47)$. The normalized frequency curve obtained from the histogram of the data and the ML estimate of the GL pdf are plotted as shown in Figure 1. The GL distribution appears to fit the data well. As a comparison with the inverse Gaussian distribution, which also has an UBT-shaped hazard rate, Figure 2 gives the plot of the hazard rates for the two distributions. There is considerable overlap between the two curves. This implies that the GL distribution is a good alternative to the inverse Gaussian in fitting lifetime data.

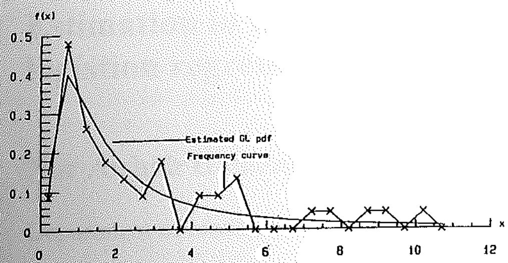

Figure 1. Normalised frequency cure and MLE of the GL pdf.

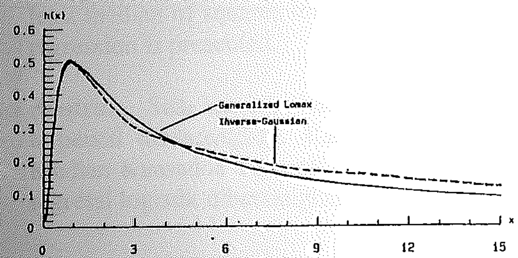

Figure 2. MLE of the hazard rate function.

References

1 Dubey, S.D. (1970). *Metrika* **16**: 27.
2 Nayak, T.K. (1987). *J. Appl. Prob.* **24**: 170.
3 Kalbfleisch, J.D. and Prentice, R.L. (1980). *The Statistical Analysis of Failure Time Data.* John Wiley, New York.
4 Turner, M.E. (1975). *Math. Biosci.* **23**: 219.
5 Lefante, J.J. and Inman, H.F. (1987). *Math. Biosci.* **83**: 167.
6 Ong, S.H. (1997). *J. Appl. Statist. Sci.* **6**: 000.
7 Chhikara, R.S. and Folks, J.L. (1977). *Technometrics* **19**: 461.
8 Glaser, R.E. (1980). *J. Am. Statist. Assoc.* **75**: 667.
9 Barlow, R.E., Marshal, A.W. and Proschan, F. (1962). *Boeing Sci. Res. Lab.* 375.
10 von Alven, W.H. (1964). *Reliability Engineering by ARINC.* Prentice Hall. Inc., New Jersey.
11 O'Neill, R. (1971). *Appl.. Statist.* **20**: 338.
12 Chambers, J.M. and Ertel, J.E. (1974). *Appl. Statist.* 250.